Sojourn times in open and closed queueing systems

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1. Literature review
2. Closed cyclic queues
3. Open tandem queues
1. Literature review

J.R. Jackson ’57
Poisson external arrivals; independent exponential service requirements; single servers; Markovian routing.
\( X_i = \# \) customers at \( Q_i \).

\[
\Pr(X_1 = n_1, \ldots, X_K = n_K) = G \prod_{i=1}^{K} \left( \frac{\Lambda_i}{\mu_i} \right)^{n_i},
\]

with \( \Lambda_i = \lambda_i + \sum_j \Lambda_j p_{ji} \) (throughput at \( Q_i \)).

**Product form** for queue lengths
Gordon & Newell ’67, Jackson ’63: closed exponential networks

\[ P(X_1 = n_1, \ldots, X_K = n_K) = G(N, K) \prod_{i=1}^{K} \left( \frac{\Lambda_i}{\mu_i} \right)^{n_i}, \]

with \( \Lambda_i = \sum_j \Lambda_j p_{ji} \)

Chandy, Muntz, Cohen, Kelly: open & closed

Nodes: exp. FCFS; PS; LCFS-PR; multiserver; IS.
sojourn times in open networks

Reich ’57, Burke ’68:
successive sojourn times of a customer are independent
(so the LST of the joint distribution has a product form ...)

Proof technique: time reversal

Fatal problem: overtaking
(so $Q_2, \ldots, Q_{K-1}$ must be single-server queues)

Walrand & Varaiya ’80:
extension to feedforward networks;
*independent* sojourn times if no loops
It is easy to find the joint sojourn time LST for $M/M/1 \to \cdot/G/1$, because the successive sojourn times of a customer in the $M/M/1$ queue and the second queue are independent, and because that second queue is an $M/G/1$ queue.

It is \textit{much} harder to find the joint sojourn time LST for $M/G/1 \to \cdot/M/1$ if $G$ is not exponential.
sojourn times in closed networks

Chow '80: cycle time in 2-node exponential network
Application: IBM computer system with level of multiprogramming $N$
$Q_1$: CPU; $Q_2$: I/O
Another application of closed cyclic queues: window flow control Reiser ’79.
Source wants to send packets at rate $\lambda$ to destination, via channels $Q_1, \ldots, Q_K$.
$\leq N$ unacknowledged packets in channels $Q_1, \ldots, Q_K$.
(or: $N$ pallets in a production system)
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$\leq N$ unacknowledged packets in channels $Q_1, \ldots, Q_K$.
(or: $N$ pallets in a production system)
$S_i = \text{sojourn time at } Q_i; \ C = S_1 + S_2.$

$$E[e^{-\omega_1 S_1 - \omega_2 S_2}] = \sum_{k=0}^{N-1} P(Z = k)E[e^{-\omega_1 S_1 - \omega_2 S_2} | Z = k]$$
$S_i$ = sojourn time at $Q_i$; $C = S_1 + S_2$.

\[
\mathbb{E}[e^{-\omega_1 S_1 - \omega_2 S_2}] = \sum_{k=0}^{N-1} \mathbb{P}(Z = k) \mathbb{E}[e^{-\omega_1 S_1 - \omega_2 S_2} | Z = k]
\]

\[
= \sum_{k=0}^{N-1} G\left(\frac{\mu_2}{\mu_1}\right)^k \mathbb{E}[e^{-\omega_1 S_1} | Z = k] \mathbb{E}[e^{-\omega_2 S_2} | S_1; Z = k]
\]
\[ S_i = \text{sojourn time at } Q_i; \ C = S_1 + S_2. \]

\[
\mathbb{E}[e^{-\omega_1 S_1 - \omega_2 S_2}] = \sum_{k=0}^{N-1} \mathbb{P}(Z = k) \mathbb{E}[e^{-\omega_1 S_1 - \omega_2 S_2} \mid Z = k]
\]

\[
= \sum_{k=0}^{N-1} G\left(\frac{\mu_2}{\mu_1}\right)^k \mathbb{E}[e^{-\omega_1 S_1} \mid Z = k] \mathbb{E}[e^{-\omega_2 S_2} \mid S_1; Z = k]
\]

\[
= \sum_{k=0}^{N-1} G\left(\frac{\mu_2}{\mu_1}\right)^k \mathbb{E}[e^{-\omega_1 S_1} \mid Z = k]\left(\frac{\mu_2}{\mu_2 + \omega_2}\right)^{N-k}
\]
\[
\mathbb{E}[e^{-\omega_1 S_1 - \omega_2 S_2}] = \sum_{k=0}^{N-1} \mathbb{P}(Z = k) \mathbb{E}[e^{-\omega_1 S_1 - \omega_2 S_2} | Z = k]
\]

\[
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\]

\[
= \sum_{k=0}^{N-1} G\left(\frac{\mu_2}{\mu_1}\right)^k \left(\frac{\mu_1}{\mu_1 + \omega_1}\right)^{k+1} \left(\frac{\mu_2}{\mu_2 + \omega_2}\right)^{N-k}.
\]
Schassberger & Daduna '83

\[ \mathbb{E}[e^{-\omega C}] = \sum_{\sum j_i = N-1} p(j_1, \ldots, j_K) \prod_{i=1}^{K} \left( \frac{\mu_i}{\mu_i + \omega} \right)^{j_i+1}, \]

with \( p(j_1, \ldots, j_K) = G(N - 1, K) \prod_{i=1}^{K} \left( \frac{1}{\mu_i} \right)^{j_i} \);

queue length distribution right after a departure (cf. the arrival theorem).
\[ E[e^{-\omega_1 S_1 - ... - \omega_K S_K}] = \sum_{\sum j_i = N - 1} p(j_1, \ldots, j_K) \prod_{i=1}^{K} \left( \frac{\mu_i}{\mu_i + \omega_i} \right)^{j_i+1}, \]
& Kelly, Konheim ’84

\[
\mathbb{E}[e^{-\omega_1 S_1 - \cdots - \omega_K S_K}] = \sum_{\sum j_i = N-1} p(j_1, \ldots, j_K) \prod_{i=1}^{K} \left( \frac{\mu_i}{\mu_i + \omega_i} \right)^{j_i+1},
\]

\[
cov(S_i, S_j) = \frac{1}{\mu_i \mu_j} \frac{1}{\mu_i + \omega_i} \text{cov}(Z_i, Z_j),
\]

\[
(S_1, \ldots, S_K) \overset{d}{=} \left( \sum_{1}^{Z_1+1} B_{1i}, \ldots, \sum_{1}^{Z_K+1} B_{Ki} \right).
\]

2. Closed cyclic queues

Known (Boxma ’83): distribution of $S_M + S_G$, and even the joint distribution of $S_M$ and $S_G$.

Unknown: distribution of $S_G + S_M$.
One can show that they do not coincide (when $G \neq M$).
Take \( G = D \) and \( N = 2 \). \( E_\mu \) denotes an \( \exp(\mu) \) r.v.

\[
S_M = \max(0, E^{(1)}_\mu - D) + E^{(2)}_\mu \quad \text{is independent of the previous } S_D.
\]

\[
S_D = \max(0, D - E^{(0)}_\mu) + D \quad \text{is negatively correlated with the previous } S_M.
\]
Recent work (& Daduna): Joint distribution of $S_G$ and $S_M$. Start from the joint distribution of number of customers $X^a_G$ and residual service time $R$ seen by an arrival at $Q_1$ of tagged customer $K$ (known $M/G/1 - N$ result).

Consider the situation at time $R$. Let $\tilde{S}_G$ denote the remaining sojourn time in $Q_1$, after $R$ (so $S_G = R + \tilde{S}_G$). Let

$$
\psi(k, h, \omega_G, \omega_M) = \mathbb{E}[e^{-\omega_G \tilde{S}_G - \omega_M S_M} | k, h]
$$

where the condition gives the numbers of customers in $Q_1$ and $Q_2$, ahead of $K$, at the start of the first new service after the arrival of $K$. 
One can write:

\[
\mathbb{E}[e^{-\omega_G S_G - \omega_M S_M}] = \mathbb{P}(X^a_G = 0)\psi(0, N - 1, \omega_G, \omega_M) \\
+ \int_{t=0}^{\infty} e^{-\omega_G t} \mathbb{P}(X^a_G = N - 1, R \in (t, t + dt))\psi(N - 2, 1, \omega_G, \omega_M) \\
+ \sum_{k=1}^{N-2} \int_{t=0}^{\infty} e^{-\omega_G t} \mathbb{P}(X^a_G = k, R \in (t, t + dt)) \times \\
\times \left\{ \sum_{l=0}^{N-k-2} e^{-\mu t} \frac{(\mu t)^l}{l!} \psi(k - 1, N - k - 1 - l + 1, \omega_G, \omega_M) \right\} \\
+ \sum_{l=N-k-1}^{\infty} e^{-\mu t} \frac{(\mu t)^l}{l!} \psi(k - 1, 1, \omega_G, \omega_M) \right\}.
\]

So if we have determined all \(\psi(k, h, \omega_G, \omega_M), k = 0, 1, \ldots, N - 2,\)  
\(h = 0, 1, \ldots, N - 1,\) we are done.
Determination of $\psi(k, h, \omega_G, \omega_M)$, $k = 0, 1, \ldots, N-2$, $h = 0, 1, \ldots, N-1$.

$\psi(0, h, \omega_G, \omega_M)$ is easy.

$$\psi(k, 0, \omega_G, \omega_M) = G^*(\omega_G)\psi(k - 1, 1, \omega_G, \omega_M).$$

$$\psi(k, h, \omega_G, \omega_M) = \int_{t=0}^{\infty} e^{-\omega_G t} \left\{ \sum_{l=0}^{h-1} e^{-\mu t} \frac{(\mu t)^l}{l!} \psi(k - 1, h - l + 1, \omega_G, \omega_M) \right\} \, dG(t).$$

Notice that we have $\psi(k - 1, \cdot, \omega_G, \omega_M)$ in all terms in the rhs.

Shorthand notation (suppress $\omega_G$ and $\omega_M$):

$$\psi(k, h) = \sum_{r=2}^{h+1} \psi(k - 1, r) a(h - r + 1) + \psi(k - 1, 1) b(h).$$
Introduce the vectors, for $k = 0, 1, \ldots, N - 1$:

$$\bar{\psi}(k) := (\psi(k, N - k - 1), \psi(k, N - k - 2), \ldots, \psi(k, 1)),$$

and the $(N - k) \times (N - k - 1)$ matrices $A(N - k), k = 1, \ldots, N - 1$:

$$
\begin{pmatrix}
  a(0) & 0 & 0 & \ldots & 0 & 0 \\
  a(1) & a(0) & 0 & \ldots & \ldots & \ldots \\
  a(2) & a(1) & a(0) & \ldots & \ldots & \ldots \\
  a(3) & a(2) & a(1) & \ldots & \ldots & \ldots \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  \vdots & \vdots & \vdots & \ddots & 0 & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots & a(0) \\
  a(N - k - 2) & a(N - k - 3) & a(N - k - 4) & \ldots & a(1) & a(0) \\
  b(N - k - 1) & b(N - k - 2) & b(N - k - 3) & \ldots & b(2) & b(1)
\end{pmatrix} \quad (2)
$$
Hence one can write

\[
\bar{\psi}(k) = \bar{\psi}(k - 1)A(N - k) = \bar{\psi}(k - 2)A(N - k + 1)A(N - k) = \ldots = \bar{\psi}(0) \prod_{j=1}^{k} A(N - j),
\]

where \(\bar{\psi}(0)\) is easily determined.

Thus all \(\psi(k, h) = \psi(k, h, \omega_G, \omega_M)\) are known, yielding

\[
E[e^{-\omega_G S_G - \omega_M S_M}].
\]
Blanc, Iasnogorodski and Nain ’88 used a boundary value approach (formulation as a Riemann-Hilbert problem) to obtain

\[ \mathbb{P}(X_G^a = j, R \in (t, t + dt), X_M^a = m) \]

as seen by arriving customer \( K \).
Knowing $\mathbb{P}(X_G^a = j, R \in (t, t + dt), X_M^a = m)$ we can write:

$$
\mathbb{E}[e^{-\omega_G S_G - \omega_M S_M}] = \sum_{m=0}^{\infty} \mathbb{P}(X_G^a = 0, X_M^a = m)\psi(0, m, \omega_G, \omega_M)
$$

$$
+ \sum_{j=1}^{\infty} \sum_{m=0}^{\infty} \int_{t=0}^{\infty} e^{-\omega_G t} \mathbb{P}(X_G^a = j, R \in (t, t + dt), X_M^a = m) \times
\times \left\{ \sum_{l=0}^{m-1} e^{-\mu t} \frac{(\mu t)^l}{l!} \psi(j - 1, m - l + 1, \omega_G, \omega_M) + \sum_{l=m}^{\infty} e^{-\mu t} \frac{(\mu t)^l}{l!} \psi(j - 1, 1, \omega_G, \omega_M) \right\}.
$$

(3)

This is the same $\psi$ function, except that it does not stop at $N - 1$. 
The recursion (1) for $\psi(k, h, \omega_G, \omega_M)$ allows us, in a straightforward way, to obtain $A(x, y, \omega_G, \omega_M) := \sum_{k=0}^{\infty} \sum_{h=1}^{\infty} x^k y^h \psi(k, h, \omega_G, \omega_M)$:

$$A(x, y, \omega_G, \omega_M) = \frac{\mu}{\mu + \omega_M} \frac{y}{1 - y} \frac{1}{y - \omega_G \mu(1 - y)} \times \left[ x \frac{G^*(\omega_G) - \frac{\omega_M}{\omega_M + \mu(1-f)} f}{1 - x G^*(\omega_G)} \left( y G^*(\omega_G) - G^*(\omega_G + \mu(1 - y)) \right) \right] + y \left( G^*(\omega_G) - \frac{\omega_M}{\omega_M + \mu(1 - y)} G^*(\omega_G + \mu(1 - y)) \right).$$

(4)

Here

$$f = f(x, \omega_G) = \mathbb{E}[x^N e^{-\omega_G P}],$$

with $P$ and $N$ the busy period and number served in $P$ in the $M/G/1$ queue $Q_1$. 
Technical problems:
(i) Blanc et al. don’t give $\mathbb{P}(X_G^a = j, R \in (t, t + dt), X_M^a = m)$ and $\mathbb{P}(X_G^a = 0, X_M^a = m)$ but their transforms;
(ii) we get the transform $A(x, y, \omega_G, \omega_M)$ of $\psi(k, h, \omega_G, \omega_M)$ w.r.t. $k$ and $h$.

Question: How to handle those problems, if you need, e.g.,

$$\sum_{m=0}^{\infty} \mathbb{P}(X_G^a = 0, X_M^a = m)\psi(0, m, \omega_G, \omega_M).$$
Consider the first term in the rhs of (3):

\[
\begin{align*}
\sum_{m=0}^{\infty} \mathbb{P}(X_G^a = 0, X_M^a = m) \psi(0, m, \omega_G, \omega_M) &= \mathbb{P}(X_G^a = 0, X_M^a = 0) \frac{G^*(\omega_G)\mu}{\mu + \omega_M} \\
+ \sum_{m=1}^{\infty} \mathbb{P}(X_G^a = 0, X_M^a = m) \psi(0, m, \omega_G, \omega_M) \\
= \mathbb{P}(X_G^a = 0, X_M^a = 0) G^*(\omega_G) \frac{\mu}{\mu + \omega_M} \\
+ \sum_{m=1}^{\infty} \mathbb{P}(X_G^a = 0, X_M^a = m) \frac{1}{2\pi i} \int_{|z|=1} \frac{\mathbb{A}(0, z, \omega_G, \omega_M)}{z^{m+1}} \, dz \\
= \mathbb{P}(X_G^a = 0, X_M^a = 0) G^*(\omega_G) \frac{\mu}{\mu + \omega_M} \\
+ \frac{1}{2\pi i} \int_{|z|=1} \frac{\mathbb{A}(0, z, \omega_G, \omega_M)}{z} \mathbb{E}\left[\left(\frac{1}{z}\right)^{X_M^a} | X_G^a = 0, X_M^a > 0\right] \, dz.
\end{align*}
\]
Here

\[ A(0, z, \omega_G, \omega_M) = \frac{\mu}{\mu + \omega_M} \frac{z}{1 - z} \left[ G^*(\omega_G) - \frac{\omega_M}{\omega_M + \mu(1 - z)} G^*(\omega_G + \mu(1 - z)) \right]. \]

\( z = 1 \): removable singularity.

\( \omega_M + \mu(1 - z) = 0 \) gives pole \( z = \frac{\mu + \omega_M}{\mu} \).

Finally, there might be poles of \( G^*(\omega_G + \mu(1 - z)) \).
Consider the contour integral \( \int_{|z|=1} \), or rather the closed contour \( C \) also involving the large circle with radius \( L \to \infty \):

\[
\frac{1}{2\pi i} \int_C \frac{A(0, z, \omega_G, \omega_M)}{z} \mathbb{E}\left[\left(\frac{1}{z}\right)^{X_M^a}(X_G^a = 0, X_M^a > 0)\right] dz.
\]

For \( |z| > 1 \), the only poles are those of \( A(0, z, \omega_G, \omega_M) \). These are \( z = \frac{\mu + \omega_M}{\mu} \), and the poles of \( G^*(\omega + \mu(1 - z)) \).

Now use Cauchy’s theorem. Calculus of residues: the integral over the closed contour equals minus the sum of the residues of all poles.
Final remarks

- Many technicalities have to be handled
- $G(x) = 1 - e^{-\alpha x}$ gives known $M/M/1 - \cdot/M/1$ results
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• Many technicalities have to be handled
• \( G(x) = 1 - e^{-\alpha x} \) gives known \( M/M/1 - \cdot /M/1 \) results
• Gideon, I wish you and Josie all the best!