The class of distributions associated with the generalized Pollaczek-Khinchine formula

Offer Kella

The Hebrew University of Jerusalem
\[ X = \{ X_t \mid t \geq 0 \} - \text{càdlàg Lévy process with no negative jumps and } X_0 = 0 \text{ a.s.} \]
\[ X = \{ X_t \mid t \geq 0 \} - \text{càdlàg Lévy process with no negative jumps and} \]
\[ X_0 = 0 \text{ a.s.} \]
Then \[ E e^{-\alpha X_t} = e^{\phi(\alpha) t}, \]
\( X = \{X_t \mid t \geq 0\} \) - càdlàг Lévy process with no negative jumps and \( X_0 = 0 \) a.s.
Then \( E e^{-\alpha X_t} = e^{\varphi(\alpha) t} \), where

\[
\varphi(\alpha) = b\alpha + \frac{\sigma^2}{2} \alpha^2 + \int_{(0,\infty)} (e^{-\alpha x} - 1 + \alpha x 1_{\{x \leq 1\}}) \nu(dx)
\]
\( X = \{ X_t \mid t \geq 0 \} \) - càdlàg Lévy process with no negative jumps and \( X_0 = 0 \) a.s.
Then \( \mathbb{E} e^{-\alpha X_t} = e^{\varphi(\alpha) t} \), where

\[
\varphi(\alpha) = b\alpha + \frac{\sigma^2}{2}\alpha^2 + \int_{(0,\infty)} \left( e^{-\alpha x} - 1 + \alpha x 1_{\{x \leq 1\}} \right) \nu(dx)
\]

\( \nu \) is the Lévy measure satisfying

\[
\int_{(0,1]} x^2 \nu(dx) < \infty \quad \text{and} \quad \nu((1,\infty)) < \infty
\]
\[
\frac{EX_t}{t} = -\varphi'(0+) = \int_{(1,\infty)} x\nu(dx) - b
\]
\[
\frac{EX_t}{t} = -\varphi'(0+) = \int_{(1,\infty)} x\nu(dx) - b \in (-\infty, \infty]
\]
Workshop in honor of Gideon - June 6-8, 2012

\[ \frac{EX_t}{t} = -\varphi'(0+) = \int_{(1,\infty)} x\nu(dx) - b \in (-\infty, \infty] \]

When \( \int_{(1,\infty)} x\nu(dx) < \infty \) then
\[ \frac{EX_t}{t} = -\varphi'(0+) = \int_{(1,\infty)} x\nu(dx) - b \in (-\infty, \infty] \]

When \( \int_{(1,\infty)} x\nu(dx) < \infty \) then

\[ \varphi(\alpha) = \mu\alpha + \frac{\sigma^2}{2}\alpha^2 + \int_{(0,\infty)} (e^{-\alpha x} - 1 + \alpha x) \nu(dx) \]
\[
\frac{EX_t}{t} = -\varphi'(0+) = \int_{(1,\infty)} x\nu(dx) - b \in (-\infty, \infty]
\]

When \( \int_{(1,\infty)} x\nu(dx) < \infty \) then

\[
\varphi(\alpha) = \mu\alpha + \frac{\sigma^2}{2}\alpha^2 + \int_{(0,\infty)} (e^{-\alpha x} - 1 + \alpha x) \nu(dx)
\]

where \( \mu = \varphi'(0) = b - \int_{(1,\infty)} x\nu(dx) > 0 \).
When $\int_{(0,1]} x \nu(dx) < \infty$ then

$$\varphi(\alpha) = c\alpha + \frac{\sigma^2}{2} \alpha^2 - \int_{(0,\infty)} (1 - e^{-\alpha x}) \nu(dx),$$

where $c = b + \int_{(0,1]} x \nu(dx)$. 
When \( \int_{(0,1]} x\nu(dx) < \infty \) then

\[
\varphi(\alpha) = c\alpha + \frac{\sigma^2}{2}\alpha^2 - \int_{(0,\infty)} (1 - e^{-\alpha x}) \nu(dx),
\]

where \( c = b + \int_{(0,1]} x\nu(dx) \).

In this case,

\[
X_t = B_t + J_t
\]

where \( B \)-Brownian motion, \( J \)-subordinator and \( B \perp J \).
Let
Let

\[ M_t = \sup_{0 \leq s \leq t} X_s \]
Let

\[ M_t = \sup_{0 \leq s \leq t} X_s \rightarrow \sup_{s \geq 0} X_s \]
Let

\[ M_t = \sup_{0 \leq s \leq t} X_s \rightarrow \sup_{s \geq 0} X_s \equiv M_\infty, \]
Let

- $M_t = \sup_{0 \leq s \leq t} X_s \rightarrow \sup_{s \geq 0} X_s \equiv M_\infty$,
- $L_t = -\inf_{0 \leq s \leq t} X_s$.

Then for every $t \geq 0$, $(W_t, L_t)$ $\sim$ $(M_t, M_t - X_t)$.
Let

- $M_t = \sup_{0 \leq s \leq t} X_s \rightarrow \sup_{s \geq 0} X_s \equiv M_\infty$,
- $L_t = -\inf_{0 \leq s \leq t} X_s$,
- $W_t = X_t + L_t$ (reflected/regulated process).
Let

\[ M_t = \sup_{0 \leq s \leq t} X_s \rightarrow \sup_{s \geq 0} X_s \equiv M_\infty, \]

\[ L_t = -\inf_{0 \leq s \leq t} X_s, \]

\[ W_t = X_t + L_t \] (reflected/regulated process).

Then for every \( t \geq 0 \)

\[ (W_t, L_t) \sim (M_t, M_t - X_t) \]
Thus

\[ P[M_\infty < \infty] = 1 \iff W_t \xrightarrow{d} M_\infty \]
Thus

\[ P[M_\infty < \infty] = 1 \iff W_t \overset{d}{\to} M_\infty \]

which is equivalent to

\[ \frac{EX_t}{t} = -\varphi'(0) < 0 \]
Thus

\[ P[M_\infty < \infty] = 1 \iff W_t \xrightarrow{d} M_\infty \]

which is equivalent to

\[ \frac{EX_t}{t} = -\varphi'(0) < 0 \iff \int_{(1,\infty)} x\nu(dx) < b \]
Thus

\[ P[M_\infty < \infty] = 1 \iff W_t \xrightarrow{d} M_\infty \]

which is equivalent to

\[ \frac{EX_t}{t} = -\varphi'(0) < 0 \iff \int_{(1,\infty)} x\nu(dx) < b \]

in particular \( \int_{(1,\infty)} x\nu(dx) < \infty. \)
In this case

\[ E e^{-\alpha W_t} \rightarrow E e^{-\alpha M_\infty} = \frac{\alpha \varphi'(0)}{\varphi(\alpha)} \]
In this case

\[ E e^{-\alpha W_t} \rightarrow E e^{-\alpha M_\infty} = \frac{\alpha \varphi'(0)}{\varphi(\alpha)} \]

‘Generalized Pollaczek-Khinchin formula’
If $\nu(0, \infty) = 0$ then

$$\varphi(\alpha) = c\alpha + \frac{\sigma^2}{2} \alpha^2$$

Hence $\varphi'(0) = c$ and if $c > 0$ then

$$\alpha \varphi'(0) = \alpha c + \alpha \frac{\sigma^2}{2} \alpha = 2c \sigma^2$$

Thus, $M_\infty \sim \exp\left(\frac{2c}{\sigma^2}\right)$
If $\nu(0, \infty) = 0$ then

$$\varphi(\alpha) = c\alpha + \frac{\sigma^2}{2}\alpha^2$$

Hence $\varphi'(0) = c$ and if $c > 0$ then

$$\frac{\alpha\varphi'(0)}{\varphi(\alpha)} = \frac{\alpha c}{\alpha c + \frac{\sigma^2}{2}\alpha^2} = \frac{2c}{\sigma^2} + \alpha$$

Thus,

$$M_\infty \sim \exp\left(\frac{2c}{\sigma^2}\right)$$
If $\nu(0, \infty) = 0$ then

$$\varphi(\alpha) = c\alpha + \frac{\sigma^2}{2}\alpha^2$$

Hence $\varphi'(0) = c$ and if $c > 0$ then

$$\frac{\alpha \varphi'(0)}{\varphi(\alpha)} = \frac{\alpha c}{\alpha c + \frac{\sigma^2}{2}\alpha^2} = \frac{2c}{\sigma^2 + \alpha}$$

Thus,

$$M_\infty \sim \exp\left(\frac{2c}{\sigma^2}\right)$$
If $\sigma^2 = 0$ then

$$
\varphi(\alpha) = c\alpha - \int_{(0,\infty)} (1 - e^{-\alpha x}) \nu(dx)
$$

$$
= c\alpha - \int_{(0,\infty)} \int_{0}^{x} \left( \alpha e^{-\alpha y} \right) dy \nu(dx)
$$

$$
= \alpha \left( c - \int_{0}^{\infty} e^{-\alpha y} \nu(y, \infty) dy \right)
$$
With

\[ \bar{\nu} = \int_{(0, \infty)} x \nu(dx) < \infty, \]
With

- $\bar{\nu} = \int_{(0,\infty)} x\nu(dx) < \infty$,
- $f_e(x) = \frac{\nu(x,\infty)}{\bar{\nu}}$, 

we have $\phi(\alpha) = \alpha c (1 - \rho \tilde{F}_e(\alpha))$ with $\phi'(0) = c (1 - \rho)$. 
With

- $\bar{\nu} = \int_{(0,\infty)} x \nu(dx) < \infty$,
- $f_e(x) = \frac{\nu(x,\infty)}{\bar{\nu}}$,
- $\tilde{F}_e(\alpha) = \int_{0}^{\infty} e^{-\alpha x} f_e(x) dx$, 

and

$\rho = \bar{\nu} c < 0$ (recall $\bar{\nu} < c$), we have

$\phi(\alpha) = \frac{\alpha c (1 - \rho \tilde{F}_e(\alpha))}{c (1 - \rho)}$. 

With

- $\bar{\nu} = \int_{(0,\infty)} x\nu(dx) < \infty$,
- $f_e(x) = \frac{\nu(x,\infty)}{\bar{\nu}}$,
- $\tilde{F}_e(\alpha) = \int_0^\infty e^{-\alpha x} f_e(x) dx$,
- $\rho = \frac{\bar{\nu}}{c} < 0$
With

- \( \bar{\nu} = \int_{(0,\infty)} x \nu(dx) < \infty \),
- \( f_e(x) = \frac{\nu(x,\infty)}{\bar{\nu}} \),
- \( \tilde{F}_e(\alpha) = \int_{0}^{\infty} e^{-\alpha x} f_e(x) dx \),
- \( \rho = \frac{\bar{\nu}}{c} < 0 \) (recall \( \bar{\nu} < c \)).
With

- $\tilde{\nu} = \int_{(0,\infty)} x \nu(dx) < \infty$,
- $f_e(x) = \frac{\nu(x,\infty)}{\tilde{\nu}}$,
- $\tilde{F}_e(\alpha) = \int_0^\infty e^{-\alpha x} f_e(x) dx$,
- $\rho = \frac{\tilde{\nu}}{c} < 0$ (recall $\tilde{\nu} < c$),

we have

$$\varphi(\alpha) = \alpha c (1 - \rho \tilde{F}_e(\alpha))$$
With

\[ \bar{\nu} = \int_{(0,\infty)} x \nu(dx) < \infty, \]
\[ f_e(x) = \frac{\nu(x,\infty)}{\bar{\nu}}, \]
\[ \tilde{F}_e(\alpha) = \int_{0}^{\infty} e^{-\alpha x} f_e(x) dx, \]
\[ \rho = \frac{\bar{\nu}}{c} < 0 \text{ (recall } \bar{\nu} < c), \]

we have

\[ \varphi(\alpha) = \alpha c (1 - \rho \tilde{F}_e(\alpha)) \]

with

\[ \varphi'(0) = c(1 - \rho) \]
Thus
\[
\frac{\alpha \varphi'(0)}{\varphi(\alpha)} = \frac{1 - \rho}{1 - \rho \tilde{F}_e(\alpha)}
\]
Thus

\[
\frac{\alpha \varphi'(0)}{\varphi(\alpha)} = \frac{1 - \rho}{1 - \rho \tilde{F}_e(\alpha)} = \sum_{n=0}^{\infty} (1 - \rho)^n \tilde{F}_e(\alpha)
\]
Thus

\[
\frac{\alpha \varphi'(0)}{\varphi(\alpha)} = \frac{1 - \rho}{1 - \rho \tilde{F}_e(\alpha)} = \sum_{n=0}^{\infty} (1 - \rho) \rho^n \tilde{F}_e(\alpha)
\]

Pollaczek-Khinchine formula,
Thus
\[
\frac{\alpha \varphi'(0)}{\varphi(\alpha)} = \frac{1 - \rho}{1 - \rho \tilde{F}_e(\alpha)} = \sum_{n=0}^{\infty} (1 - \rho) \rho^n \tilde{F}_e(\alpha)
\]

Pollaczek-Khinchine formula,
Holds without change for general subordinators! (not only compound Poisson)
The case $B + J$,
The case $B + J$, i.e., $\int_{(0,1]} x \nu(dx) < \infty$. 
The case $B + J$, i.e., $\int_{(0,1]} x \nu(dx) < \infty$.

$$\varphi(\alpha) = c \alpha \left( 1 + \frac{\sigma^2}{2c} \alpha - \rho \tilde{F}_e(\alpha) \right)$$
The case $B + J$, i.e., $\int_{(0,1]} xn(dx) < \infty$.

$$\varphi(\alpha) = c\alpha \left( 1 + \frac{\sigma^2}{2c} \alpha - \rho \tilde{F}_e(\alpha) \right)$$

so

$$\varphi'(0) = c(1 - \rho)$$
Hence, with $\lambda = 2c/\sigma^2$, 

$$\frac{\alpha \varphi'(0)}{\varphi(\alpha)} = \frac{1 - \rho}{1 + \alpha/\lambda - \rho \tilde{F}_e(\alpha)}$$
Hence, with $\lambda = 2c/\sigma^2$,

$$\frac{\alpha \varphi'(0)}{\varphi(\alpha)} = \frac{1 - \rho}{1 + \alpha/\lambda - \rho \tilde{F}_e(\alpha)} = \frac{1}{1 + \alpha/\lambda} \cdot \frac{1 - \rho}{1 - \rho \frac{1}{1+\alpha/\lambda} \tilde{F}_e(\alpha)}$$
Hence, with $\lambda = 2c/\sigma^2$,

$$\frac{\alpha \varphi'(0)}{\varphi(\alpha)} = \frac{1 - \rho}{1 + \alpha/\lambda - \rho \tilde{F}_e(\alpha)} = \frac{1}{1 + \alpha/\lambda} \cdot \frac{1 - \rho}{1 - \rho \frac{1}{1 + \alpha/\lambda} \tilde{F}_e(\alpha)}$$

or

$$\frac{\alpha \varphi'(0)}{\varphi(\alpha)} = \frac{\lambda}{\lambda + \alpha} \sum_{n=0}^{\infty} (1 - \rho) \rho^n \left( \frac{\lambda}{\lambda + \alpha} \tilde{F}_e(\alpha) \right)^n$$
Theorem: Assume $\sigma^2 > 0$, $0 < \bar{\nu} = \int_{(0,\infty)} x\nu(dx) < c$. 
**Theorem:** Assume \( \sigma^2 > 0, \ 0 < \bar{\nu} = \int_{(0,\infty)} x \nu(dx) < c. \)

Let \( N, X_0, X_1, \ldots, Y_1, Y_2, \ldots \) be independent with
**Theorem:** Assume $\sigma^2 > 0$, $0 < \bar{\nu} = \int_{(0,\infty)} x\nu(dx) < c$. Let $N, X_0, X_1, \ldots, Y_1, Y_2, \ldots$ be independent with

- $X_i \sim \exp\left(\frac{2c}{\sigma^2}\right)$,
**Theorem:** Assume $\sigma^2 > 0$, $0 < \bar{\nu} = \int_{(0,\infty)} x \nu(dx) < c$. Let $N, X_0, X_1, \ldots, Y_1, Y_2, \ldots$ be independent with

- $X_i \sim \text{exp} \left( \frac{2c}{\sigma^2} \right)$,

- $Y_i$ with density $f_e(x) = \frac{\nu(x,\infty)}{\bar{\nu}}$,
**Theorem:** Assume $\sigma^2 > 0$, $0 < \bar{\nu} = \int_{(0,\infty)} x \nu(dx) < c$. Let $N, X_0, X_1, \ldots, Y_1, Y_2, \ldots$ be independent with

- $X_i \sim \text{exp} \left( \frac{2c}{\sigma^2} \right)$,
- $Y_i$ with density $f_e(x) = \frac{\nu(x,\infty)}{\bar{\nu}}$,
- $N + 1 \sim G((1 - \rho))$. 
**Theorem:** Assume $\sigma^2 > 0$, $0 < \bar{\nu} = \int_{(0,\infty)} x\nu(dx) < c$. Let $N, X_0, X_1, \ldots, Y_1, Y_2, \ldots$ be independent with

- $X_i \sim \text{exp}\left(\frac{2c}{\sigma^2}\right)$,
- $Y_i$ with density $f_e(x) = \frac{\nu(x,\infty)}{\bar{\nu}}$,
- $N + 1 \sim G((1 - \rho))$.

Then

$$M_\infty \sim X_0 + \sum_{i=1}^{N} (X_i + Y_i)$$

where an empty sum is zero.
Theorem: Let $N, X_0, X_1, \ldots, Y_1, Y_2, \ldots$ be independent with $X_i \sim \exp(\lambda)$, $Y_i$ has nonincreasing density, $N + 1 \sim G(p)$, and $Z \sim X_0 + \sum_{i=1}^{\infty} (X_i + Y_i)$, where an empty sum is zero. Then there is a Lévy process which is an independent sum of a Brownian motion and a subordinator for which $Z \sim M_{\infty}$. This Lévy process is unique up to change of time scale.
Theorem: Let $N, X_0, X_1, \ldots, Y_1, Y_2, \ldots$ be independent with
- $X_i \sim \exp(\lambda)$,
**Theorem:** Let $N, X_0, X_1, \ldots, Y_1, Y_2, \ldots$ be independent with

- $X_i \sim \exp(\lambda),$
- $Y_i$ has nonincreasing density,
Theorem: Let $N, X_0, X_1, \ldots, Y_1, Y_2, \ldots$ be independent with

- $X_i \sim \exp(\lambda)$,
- $Y_i$ has nonincreasing density,
- $N + 1 \sim G(p)$.
**Theorem:** Let $N, X_0, X_1, \ldots, Y_1, Y_2, \ldots$ be independent with

- $X_i \sim \exp(\lambda)$,
- $Y_i$ has nonincreasing density,
- $N + 1 \sim G(p)$.

and

$$Z \sim X_0 + \sum_{i=1}^N (X_i + Y_i)$$

where an empty sum is zero.
Theorem: Let $N, X_0, X_1, \ldots, Y_1, Y_2, \ldots$ be independent with
- $X_i \sim \exp(\lambda)$,
- $Y_i$ has nonincreasing density,
- $N + 1 \sim G(p)$.

and

$$Z \sim X_0 + \sum_{i=1}^{N} (X_i + Y_i)$$

where an empty sum is zero.

Then there is a Lévy process which is an independent sum of a Brownian motion and a subordinator for which

$$Z \sim M_\infty$$
**Theorem:** Let $N, X_0, X_1, \ldots, Y_1, Y_2, \ldots$ be independent with
- $X_i \sim \text{exp}(\lambda)$,
- $Y_i$ has nonincreasing density,
- $N + 1 \sim G(p)$.

and

$$Z \sim X_0 + \sum_{i=1}^{N} (X_i + Y_i)$$

where an empty sum is zero.

Then there is a Lévy process which is an independent sum of a Brownian motion and a subordinator for which

$$Z \sim M_{\infty}$$

This Lévy process is unique up to change of time scale.
Observations:

- $\sigma^2 = 0 \Rightarrow \text{PK.}$
Observations:

- $\sigma^2 = 0 \Rightarrow$ PK.
- $\bar{\nu} = 0 \Rightarrow$ exponential.
Observations:

- \( \sigma^2 = 0 \Rightarrow \) PK.
- \( \bar{\nu} = 0 \Rightarrow \) exponential.
- Exponential and compound geometric are infinitely divisible. Thus, so is the distribution of

\[
X_0 + \sum_{i=1}^{N} (X_i + Y_i)
\]
Observations:

- $\sigma^2 = 0 \Rightarrow \text{PK}.$
- $\bar{\nu} = 0 \Rightarrow \text{exponential}.$
- Exponential and compound geometric are infinitely divisible. Thus, so is the distribution of

$$X_0 + \sum_{i=1}^{N} (X_i + Y_i)$$

- If $J$ is compound Poisson with phase-type distributed jumps then

$$X_0 + \sum_{i=1}^{N} (X_i + Y_i)$$

has a phase-type distribution.
When \( \int_{(0,1]} x \nu(dx) = \infty \)
When \( \int_{(0,1]} x\nu(dx) = \infty \) \( (X \not
sim B + J) \),
When $\int_{(0,1]} x\nu(dx) = \infty \ (X \not\sim B + J)$, take for $0 < \epsilon \leq 1$

$$\nu_\epsilon(A) = \nu(A \cap (\epsilon, \infty))$$
When $\int_{(0,1]} x \nu(dx) = \infty$ ($X \not\sim B + J$), take for $0 < \epsilon \leq 1$

$$\nu_{\epsilon}(A) = \nu(A \cap (\epsilon, \infty))$$

so that

$$\varphi_{\epsilon}(\alpha) = b\alpha + \frac{\sigma^2}{2} \alpha^2 + \int_{(0,\infty)} (e^{-\alpha x} - 1 + \alpha x 1_{(0,1]}(x)) \nu_{\epsilon}(dx)$$

$$= b\alpha + \frac{\sigma^2}{2} \alpha^2 + \int_{(\epsilon,\infty)} (e^{-\alpha x} - 1 + \alpha x 1_{(0,1]}(x)) \nu(dx)$$
Then $\varphi_\epsilon(\alpha) \to \varphi(\alpha)$ as $\epsilon \downarrow 0,$
Then $\varphi_\epsilon(\alpha) \rightarrow \varphi(\alpha)$ as $\epsilon \downarrow 0$, while $\varphi'_\epsilon(0) = \varphi'(0)$. 


Then \( \varphi_\epsilon(\alpha) \to \varphi(\alpha) \) as \( \epsilon \downarrow 0 \),
while \( \varphi'_\epsilon(0) = \varphi'(0) \). Thus

\[
\frac{\varphi'_\epsilon(0)\alpha}{\varphi_\epsilon(\alpha)} \to \frac{\varphi'(0)\alpha}{\varphi(\alpha)}
\]
**Theorem:** A random variable has a generalized PK-LST iff its distribution is in the closure of the distributions of random variables of the form

\[ X_0 + \sum_{n=1}^{N} (X_n + Y_n) \]

satisfying the requirements of the previous theorem.
**Theorem:** A random variable has a generalized PK-LST iff its distribution is in the closure of the distributions of random variables of the form

\[ X_0 + \sum_{n=1}^{N} (X_n + Y_n) \]

satisfying the requirements of the previous theorem.

**Comment:** When there are no negative jumps, an elementary proof that $M_{\infty}$ has an infinitely divisible distribution.
VERY BEST WISHES DEAR GIDEON